On the pressure induced by the boundary layer on a flat plate in shear flow

By ALAR TOOMRE
Department of Mathematics, Massachusetts Institute of Technology
AND NICHOLAS ROTT
Department of Engineering, University of California at Los Angeles

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The problem solved is that of the interaction between a laminar boundary layer on a semi-infinite flat plate and an oncoming shear flow of finite lateral dimensions bounded by uniform irrotational flow extending to infinity. The pressures along the plate and upstream of the same are deduced (to a linearized approximation) in the form of a Fourier integral based on the solution of a simpler periodic flow problem. It is found that while the assumption of an infinite, uniform shear flow gives asymptotically correct interaction pressure gradients on the plate near the leading edge, the pressure level even there (compared to upstream infinity) is strongly influenced by the boundedness of the external shear. At distances from the leading edge which are large compared to the lateral extent of the shear flow, the pressure gradients along the plate are shown to be vanishingly smaller than in the infinite shear case.

1. Introduction

The problem of the development of a boundary layer on a semi-infinite flat plate submerged in an incompressible, parallel shear flow may at first appear to be no more than a routine extension from the standard case where the undisturbed stream is not sheared at all. It has become known, however, that a satisfactory solution of this problem, even to that degree of approximation in which the effect of the outside shear is taken into account only for the first time, demands more than a mere reworking of the ordinary boundary-layer theory.

For one thing, since any sort of a boundary layer on the plate effectively constitutes an obstacle of finite width placed in the sheared stream, one must in general expect some cross-flow to occur upstream of the plate, with the result that the total pressure—and, consequently, the downstream velocity—of the fluid arriving in the proximity of the leading edge will not be exactly the same as that found at a large distance directly ahead of the plate. This complication is rather analogous to the 'displacement effect' of a Pitot tube placed in a shear flow (e.g. Hall 1956).

The other, and in a sense more immediate, complication caused by the free-stream shear is one that was first pointed out by Li (1956) for the special case of an oncoming parallel shear flow of infinite extent but of constant vorticity.

Not only does the usual Blasius profile have to be so modified as to conform to the prescribed vorticity away from the plate, but one has also to reckon with a certain streamwise pressure gradient stemming from an interaction of that boundary layer with the external flow.

The present paper will deal only with the latter point, partly because the first complication may almost certainly be divorced from the analysis of the boundary layer itself through a redefinition of the relevant free-stream velocity as that currently found near the leading edge; besides, that effect would not even arise if the velocity profile were symmetric about the plate. The main reason for this emphasis, however, is that the induced pressure gradient calculated by Li

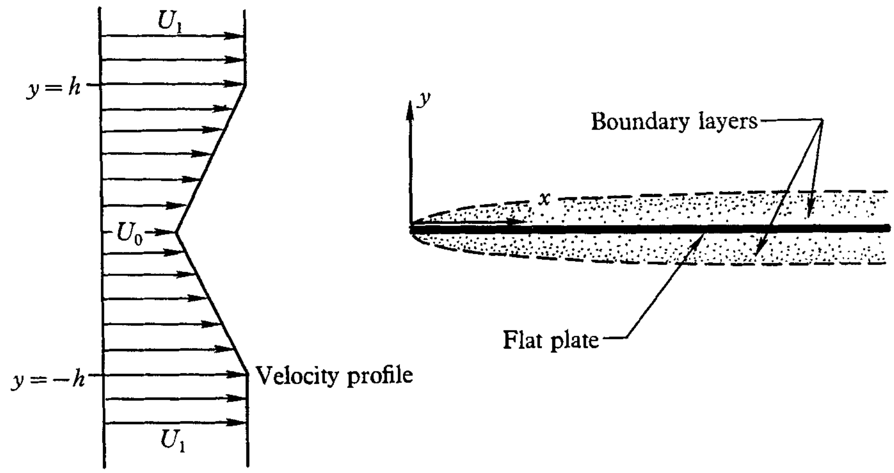

FIGURE 1. Flow situation.

seems still to be the subject of some controversy. Possibly because Li's reasoning was not made sufficiently explicit, the very notion of such a pressure gradient was soon challenged by Glauert (1957). Later, Murray (1961) proved that the pressure gradient proposed by Li would indeed occur provided the uniformly sheared flow extended laterally to infinity. However, there has as yet been no similar confirmation in any more realistic situation where the sheared region of the oncoming flow was limited in some manner; Glauert (1962) has therefore again questioned the relevance of Li and Murray's pressure gradient to problems arising in practice.

Our objective here will be to clarify this interesting, if somewhat academic, question through an analysis of the flow situation pictured in figure 1. In this example, the oncoming shear flow, $v_{x_0} = U_0 + A|y|$, extends only to a lateral distance h from the plate and is bounded on either side by an irrotational flow of speed $v_{x_0} = U_1 = U_0 + Ah$; the assumption of symmetry has obviously been made to avoid the possibility of cross-flow already mentioned above. We shall be particularly interested in determining the pressure gradient along the plate which results from the development of the boundary layer. Before we state the conclusions, though, a few words about the method are relevant. From the point of view of boundary-layer theory, the influence of external vorticity must properly be classified as a 'second-order' effect when the inverse square root of the Reynolds number is taken as the expansion parameter. It is now well understood (Van Dyke 1962) that it would be inadequate to confine one's attention to the boundary layer alone in seeking to estimate this effect. Instead, one must also consider how the boundary layer affects the outer flow, whose deflexion can be the only source of the second-order streamwise pressure gradients on a flat plate. (Kuo 1953 has made it clear that it just happens that the resulting interaction pressure is anomalously zero, for a semi-infinite flat plate in an ordinary irrotational stream, but the same need not be true in the presence of free-stream vorticity.) To determine this second-order effect in the current problem, it would in principle appear safest to apply the expansion procedure developed by Van Dyke (1962), which guarantees a systematic fitting between the inner (viscous) and the outer (inviscid) flow solutions. However, the present work will not make use of the full formalism of this very general method. Instead, we will endeavour to show how the induced pressure field can be estimated by considering only the immediate influence of the 'first-order' boundary-layer displacement thickness upon the external flow.

It will be assumed here that the approaching stream is only weakly sheared in comparison with the typical vorticity found in the boundary layer; this presupposes that we are concerned only with locations not too far downstream, so that the latter will not have had the opportunity to decrease sufficiently to have become comparable with the former. Thus, to a first order, the boundary layer will still be of the standard Blasius type, and the velocity v_y normal to the plate just outside this layer will behave as $x^{-\frac{1}{2}}$. The 'outer' flow problem may then be formulated as the effect of a source distribution along the plate, of the strength $Cx^{-\frac{1}{2}}$, upon an essentially inviscid outer shear flow; alternatively, the sources can be imagined replaced by a slender parabolic body. It is also because of this weak shear assumption that the idealized discontinuities of vorticity at $y = 0, \pm h$ in the original flow in figure 1 can be shown to cause no serious error; it may be estimated that any action of the viscosity to round them off results in a flux deficiency (or excess) that is smaller than that arising near the plate by approximately the ratio of the free-stream shear to the boundary-layer vorticity.

From the present calculations, it will be found that for distances from the leading edge which are small compared with h, the pressure gradient, and hence the augmented shear stress, are indeed those which were first calculated by Li. This is not too surprising, since for a small enough region near $x = y = 0$ the outer shear flow ought to appear unbounded; however, quantitatively, this region of approximate validity of the Li–Murray pressure gradient turns out to be remarkably small. On the other hand, for distances downstream from the leading edge which are large compared to h (but small enough so that the boundary-layer vorticity is still appreciably larger than that of the outer flow), it will be seen that the pressure gradient has decreased to a vanishingly small multiple of the Li–Murray value, thus confirming Glauert's suspicions.

An effect not discussed previously will also be discovered to be a consequence of the finiteness of the oncoming shear region. Despite the fact that the symmetry of the problem has precluded any lateral displacement of the central streamline, we will see that the magnitude of the free-stream velocity just behind the leading edge of the plate differs appreciably from the value far (that is to say, many widths of the shear layer) upstream along $y = 0$. What is more, this difference will be seen to increase without bound with increasing width of the shear flow (keeping vorticity constant), although the speed difference between the vicinity of the leading edge and any point a *fixed* distance upstream on the dividing streamline tends to zero in the same limit. The peculiar way in which the limiting case of an infinite outer shear flow is thus approached, along with the fact that the present effect seems comparable in importance to the aforementioned 'vorticity' effect, points to the danger involved in applying the results of an unbounded shear flow analysis to an actual flow situation.

2. Shear flow bounded by a uniform stream

The mathematical problem is now well established: Considering only the region $y \geqslant 0$ in figure 1 (because of symmetry), and given that

$$v_y(x < 0, y = 0) = 0 \quad \text{and} \quad v_y(x > 0, y = 0) = Cx^{-\frac{1}{2}} \tag{1}$$

(where C is a small positive constant), and that the disturbance velocities should tend to zero at large distances, we wish to determine, implicitly, the full inviscid flow field, and in particular, the pressures and the streamwise pressure gradient near $y = 0$.

The technique of solution to be employed here relies on the fact that the cross-flow velocity from equation (1) may be described equally accurately as

$$v_y(x, 0) = (2\pi)^{-\frac{1}{2}} C \int_0^\infty k^{-\frac{1}{2}} (\sin kx + \cos kx)\, dk \tag{2}$$

for all negative and positive x (e.g. Erdelyi 1954, pp. 10, 68). Thanks to this Fourier representation, our immediate task reduces to the question of how the oncoming, supposedly inviscid flow would react to a simple normal velocity like

$$v_y(x, 0) = \sin kx \tag{3}$$

imposed at its boundary.

To answer the latter question, we introduce two disturbance stream functions, ψ_1 and ψ_2. The first of these will describe the total velocity components in the region on the shear flow as

$$v_x = U_0 + Ay + \frac{\partial \psi_1}{\partial y}, \quad v_y = -\frac{\partial \psi_1}{\partial x}, \tag{4}$$

whereas the other will denote the same in the potential flow region by

$$v_x = U_1 + \frac{\partial \psi_2}{\partial y}, \quad v_y = -\frac{\partial \psi_2}{\partial x}. \tag{5}$$

Since the vorticity in either part of the flow is—and remains—constant, both of these disturbance stream functions must separately obey the Laplace equation. The forms of ψ_1 and ψ_2 with vanishing disturbance velocities at $y \to \infty$ and corresponding to the velocity of equation (3), may therefore be surmised as

$$\left.\begin{aligned}\psi_1(x,y) &= (k^{-1}\cosh ky + C_1 \sinh ky)\ \cos kx,\\ \psi_2(x,y) &= C_2 e^{-ky}\cos kx,\end{aligned}\right\} \tag{6}$$

where only the constants C_1 and C_2 remain to be determined.

We shall ascertain the values of these constants from the physical requirement that (*a*) the normal velocity, v_y, and (*b*) the streamwise pressure gradient should be the same on both sides of the interface separating the shear and potential flow regions. Strictly speaking, such a matching ought to be carried out at the (unknown) displaced location of that boundary; however, we now presume that the disturbance is infinitesimal, and so this joining shall instead be performed along $y = h$. There, by matching the y-velocities, we find from equations (6) that

$$v_y = -(\cosh kh + kC_1 \sinh kh)\sin kx = -kC_2 e^{-kh}\sin kx, \tag{7}$$

whereas the x-momentum equation gives, in linearized form,

$$\begin{aligned}-\frac{1}{\rho}\frac{\partial p}{\partial x} &= v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\\ &= -kU_1(\sinh kh + kC_1 \cosh kh)\sin kx\\ &\qquad + A(\cosh kh + kC_1 \sinh kh)\sin kx\\ &= kU_1 kC_2 e^{-kh}\sin kx.\end{aligned} \tag{8}$$

Solving equations (7) and (8), we obtain

$$kC_1 = -\frac{kU_1 e^{kh} - A\cosh kh}{kU_1 e^{kh} - A\sinh kh}, \tag{9}$$

and a rather similar expression for kC_2. These, together with equation (6), provide a full description of the disturbance velocity field corresponding to the stipulated normal velocity of equation (3).

From elementary solutions like this one, it should now be possible to synthesize the entire flow field associated with the imposed velocities of equations (1) or (2). However, our predominant interest here lies only in the pressures developed on or near the x-axis. Accordingly, we deduce from the x- and y-momentum equations and equations (6) that the simple flow discussed above implies the following pressures along the line $y = 0$:

$$\begin{aligned}p(x,0) &= -\rho\int\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right)_{y=0} dx + \text{const.}\\ &= -\rho U_0(v_x - U_0)_{y=0} - \rho A\int (v_y)_{y=0}\,dx + \text{const.}\\ &= \rho[(A/k) - U_0 kC_1]\cos kx,\end{aligned} \tag{10}$$

where, it should be noted, the constant of integration has finally been so selected as to imply a vanishing pressure as $y \to \infty$. In a similar manner, the pressure corresponding to an imposed velocity

$$v_y(x,0) = \cos kx \tag{11}$$

may be shown to differ from that of equation (10) only through the factor $\cos kx$ being replaced by $-\sin kx$. Hence, by compounding these elementary pressures in the same manner as the velocity contributions were weighted in the integral in equation (2), the pressure along $y = 0$, resulting from the interaction of the basic flow with the normal velocities of equations (1) or (2), may be calculated as

$$p(x,0) = (2\pi)^{-\frac{1}{2}}\rho AC\int_0^\infty k^{-\frac{3}{2}}\,G(kh,q)\,(\cos kx - \sin kx)\,dk, \tag{12}$$

where
$$G(kh,q) = (k/A)\,[(A/k) - U_0 kC_1] = 1 + \frac{1-q}{q}\,kh\,\frac{kh\,e^{kh} - q\cosh kh}{kh\,e^{kh} - q\sinh kh} \tag{13}$$

and
$$q = (U_1 - U_0)/U_1 = Ah/U_1. \tag{14}$$

As a partial check, we note that

$$G(kh,q) \sim kh/q(1-q) \quad \text{as} \quad kh \to 0$$

and that
$$G(kh,q) \sim (1-q)\,kh/q \quad \text{as} \quad kh \to \infty;$$

hence, the integral in equation (12) does in fact converge for all values of x, except zero.

Nevertheless, equation (12) is of a form which makes its interpretation awkward in the limit as the shear, A, and hence the parameter q, tend to zero. To render it more meaningful, we introduce a new function

$$H(kh,q) = 1 - (1-q)\,e^{-kh}\,[e^{kh} - q\sinh kh/kh]^{-1} \tag{15}$$

such that
$$G(kh,q) = (1-q)\,kh/q + H(kh,q) = (kU_1/A) + H(kh,q). \tag{16}$$

Considering that equations (1) and (2) have already implied that

$$(2\pi)^{-\frac{1}{2}}\int_0^\infty k^{-\frac{1}{2}}\,(\cos kx - \sin kx)\,dk$$

equals $(-x)^{-\frac{1}{2}}$ when $x < 0$, but vanishes for all positive x, the pressure equation may then be rewritten as

$$p(x,0) = p_0(x,0) + p_v(x,0), \tag{17}$$

where
$$p_0(x,0) = \rho U_0 C(-x)^{-\frac{1}{2}}\,(\tfrac{1}{2} - \tfrac{1}{2}\operatorname{sgn} x), \tag{18}$$

and
$$p_v(x,0) = (2\pi)^{-\frac{1}{2}}\,\rho AC\int_0^\infty k^{-\frac{3}{2}}\,H(kh,q)\,(\cos kx - \sin kx)\,dk. \tag{19}$$

The part of the second-order pressure that would arise even in the absence of shear, namely $\rho U_0 C(-x)^{-\frac{1}{2}}$ for negative x, and none at all for positive x, is given by equation (18). The quantity p_v in equation (19) clearly represents the vorticity-dependent part of the pressure, since $H(kh,q)$ remains bounded for all kh as $q \to 0$, so that p_v vanishes for $A \to 0$.

3. Unbounded shear flow

Before we proceed to discuss in detail the implications of the last three equations, it is probably best to digress briefly at this point and to rederive the Li–Murray pressure gradient as a basis for comparison. Accordingly, let us imagine that $h = \infty$ from the outset in the situation of figure 1, and let us again seek a disturbance stream function which (*a*) satisfies the Laplace equation, (*b*) exhibits the velocities of equation (1) along $y = 0$, and (*c*) involves vanishingly small disturbance velocities at large disturbances.

It is readily found that the only function that meets all those requirements is

$$\psi = -2Cr^{\frac{1}{2}}\cos\left(\tfrac{1}{2}\theta\right), \tag{20}$$

where r and θ are the usual polar co-ordinates such that $x = r\cos\theta$ and $y = r\sin\theta$, From this it follows that the x-component of the inviscid disturbance velocity,

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta}\cos\theta + \frac{\partial\psi}{\partial r}\sin\theta = -Cr^{-\frac{1}{2}}\sin\left(\tfrac{1}{2}\theta\right), \tag{21}$$

vanishes as $\theta \to 0$. Consequently, the correct boundary-layer asymptotic condition, to this order of approximation and for large values of the boundary-layer co-ordinate $y(U_0/\nu x)^{\frac{1}{2}}$, indeed is Li's condition

$$v_x \sim U_0 + Ay. \tag{22}$$

We would particularly like to emphasize that the last result is by no means trivial. It states that there is no transversal shift of the velocity profile, in spite of the fact that the fluid elements passing just outside the boundary layer have obviously been deflected from their original paths by the amount of the displacement thickness. Understandably, Glauert (and at first Li himself 1955) intuitively assumed that the particles would tend to preserve their x-velocities even as they were shunted sideways; this, however, turns out not to be the case near the plate in an infinite shear flow, and a pressure gradient appears instead. That gradient is given by

$$\frac{\partial p}{\partial x} = -\rho\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\rho A v_y = -\rho A C x^{-\frac{1}{2}}, \tag{23}$$

which is the same as that described in Li's note (1956) and later confirmed by Murray (1961). On the other hand, we also observe from equations (21) and (23) that the pressure along $\theta = \pi$ equals $\rho U_0 C(-x)^{-\frac{1}{2}}$, provided the pressure at upstream infinity is regarded as zero; therefore, the vorticity-dependent part of the pressure gradient apparently vanishes identically for $x < 0, y = 0$ when the shear extends to infinity.

4. Comparison of results

Returning now to our previous example of the bounded shear flow, we deduce from equation (19) that the pressure gradient there for $x > 0$, $y = 0$ is

$$\partial p/\partial x = -\rho A C x^{-\frac{1}{2}} F(x/h, q), \tag{24}$$

the multiplier of the part we now recognize as the Li–Murray result being

$$F(x/h, q) = (2\pi)^{-\frac{1}{2}}\int_0^\infty \xi^{-\frac{1}{2}}\, H(\xi h/x, q)\,(\sin\xi + \cos\xi)\, d\xi. \tag{25}$$

Since it may be seen from equation (15) that $H(\xi h/x, q)$ approaches unity as $x \to 0$, we find that

$$\lim_{x/h \to \infty} F(x/h, q) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \xi^{-\frac{1}{2}} (\sin \xi + \cos \xi)\, d\xi = 1. \tag{26}$$

On the other hand,

$$\lim_{x/h \to \infty} F(x/h, q) = 0, \tag{27}$$

due to $H(\xi h/x, q)$ tending to zero as $x \to \infty$. Thus, we conclude that the Li–Murray pressure gradient is approached in the downstream vicinity of the leading edge, but certainly not far down the plate.

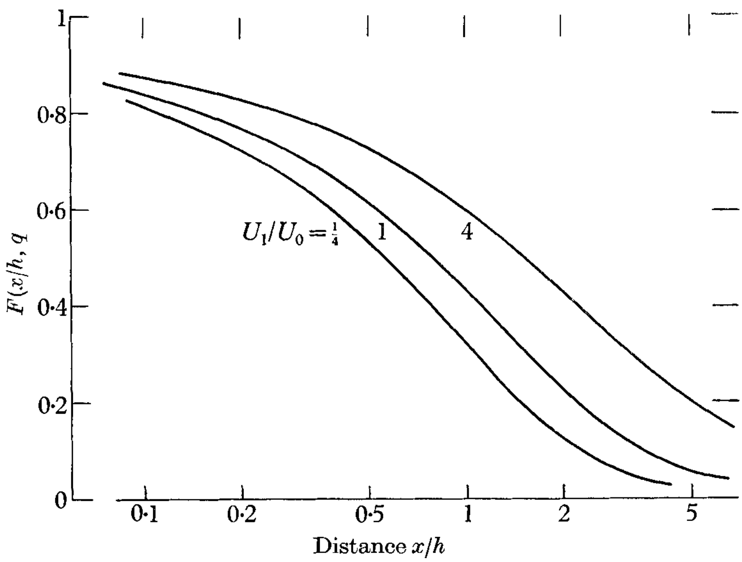

FIGURE 2. Pressure gradient along the plate, expressed as a multiple of the Li–Murray result.

The intermediate behaviour of $F(x/h, q)$ is somewhat more difficult to discern, except when the speed variation across the shear layer is very small, or $q \simeq 0$. In that case, since

$$H(\xi h/x, 0) = 1 - \exp(-2\xi h/x), \tag{28}$$

it happens that the integral in equation (24) may be evaluated explicitly to give (e.g. Erdelyi 1954, pp. 10, 14, 68, 72)

$$F(x/h, q \simeq 0) \simeq 1 - (1/2s)\,[(s-z)^{\frac{1}{2}} + (s+z)^{\frac{1}{2}}], \tag{29}$$

where

$$z = 2h/x, \quad s = (1+z^2)^{\frac{1}{2}}. \tag{30}$$

This function has been plotted against $\log(x/h)$ in figure 2, along with two similar multipliers for the cases $q = \frac{3}{4}$ and $q = -3$ (or $U_1/U_0 = 4$ and $\frac{1}{4}$, respectively) obtained by numerical integration of equation (25).

Figure 2 shows the Li–Murray limit to be approached remarkably slowly as $x/h \to 0$; for instance, when x/h is as small as one-tenth, the Li–Murray result still appears to overestimate the pressure gradient by about 15 %. From this we can only surmise that the pressure gradient must be quite sensitive to conditions at large distances. By contrast, the approach at the other extreme to the zero pressure gradient postulated by Glauert seems to be considerably more rapid, as exemplified by the fact that

$$F(x/h, q = 0) \sim 1{\cdot}5(h/x)^2 \quad \text{as} \quad x/h \to \infty.$$

Proceeding to the discussion of the vorticity-dependent pressure, p_v, itself, we conclude both on physical grounds and from the increasingly oscillatory behaviour of the integrand in equation (19) as $|x| \to \infty$, that the pressures exceedingly far upstream and downstream must in fact be *equal*. At all other locations along the x-axis, p_v can be shown to be non-negative, reaching its maximum at $x = 0$ with the value

$$p_v(0,0) = (2\pi)^{-\frac{1}{2}}\rho AC \int_0^\infty k^{-\frac{3}{2}} H(kh, q)\, dk. \tag{31}$$

For large h (but a fixed value of A), that maximum behaves roughly as $h^{\frac{1}{2}}$, as may be estimated from the full pressure drop along the plate, assuming the Li–Murray pressure gradient for $0 < x < h$, and none at all for $x > h$. The same increase without bounds may alternatively be deduced from the behaviour of $H(kh, q)$, which becomes virtually equal to unity in that limit for all but the smallest values of k.

It might be asked how the last conclusion is to be reconciled with the infinite shear flow result of § 3, which indicated that the gradient of the vorticity-dependent pressure vanished everywhere along the half-axis $x < 0$, $y = 0$, thereby implying $p_v(0,0)$ should be the same as the pressure far upstream. To help answer this, figure 3 shows the full x-behaviour of p_v along $y = 0$ for several representative velocity ratios, U_1/U_0, as determined from repeated numerical integrations of the integral in equation (19). The curves shown all refer to flows having the same basic shear, A, and central speed, U_0; however, their values of h are necessarily different. The abscissa in figure 3 is the streamwise distance x, normalized with respect to a length that equals the height h_2 for which the velocity in the shear layer (or in the linearly extrapolated shear layer, as the case may be) equals twice the central speed U_0. All pressures have been expressed as multiples of $\rho ACh_2^{\frac{1}{2}}$.

Figure 3 makes it clear that the curious unbounded increase of $p_v(0,0)$ with h is not anomalous, but merely reflects a gradual growth of that pressure ahead of the plate; this growth is markedly slower than the drop beyond $x = 0$. (We must remember that the singular, $\rho U_0 C(-x)^{-\frac{1}{2}}$ behaviour of the part of the pressure which does not depend on A, but which would usually swamp p_v for $x < 0$, has been excluded from this diagram.) We observe that not only does the region in which p_v remains approximately equal to $p_v(0, 0)$ apparently extend further and further upstream as $h \to \infty$, but even that $\partial p_v/\partial x$ at $x = 0^-$ gives the impression (that can be verified from equation (19)) of vanishing in the same limit. Thus, there is no obvious contradiction between the infinite shear flow result and this one; however, we have now seen in what limited sense only may the pressure or the velocity at the leading edge be equated in that case with the values exceedingly far upstream. It is interesting also to note that $(\partial p_v/\partial x)_{x=0^-}$ is obviously non-zero for the moderate values of h shown; this means that the only prediction of the infinite shear theory which has been proven valid near the leading edge in the case of a finite shear flow is the leading (singular) term of the pressure gradient for $x > 0$, plus, of course, $p_0(x, 0)$ as defined in equation (18).

Finally, as far as second-order boundary-layer theory is concerned, it should be emphasized that this discussion also indicates that the proper free-stream velocity to be employed at $x = 0^+$ when the shear does not extend to infinity is not exactly U_0 but U_0 minus $p_v(0,0)/\rho U_0$. Although usually quite small, this

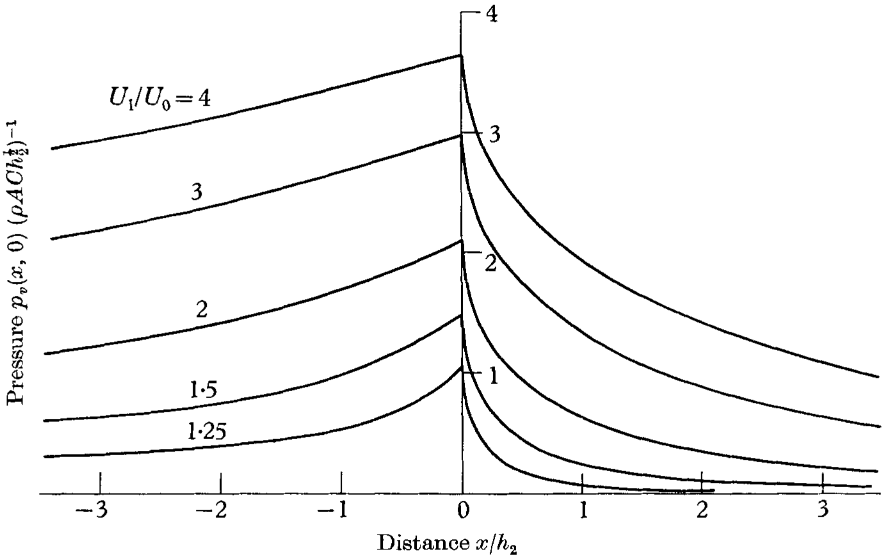

FIGURE 3. Vorticity-dependent part of the pressure upstream of and along the plate.

velocity change is clearly comparable in magnitude to that resulting from any 'vorticity-induced' pressure gradients along the plate and must be considered whenever the latter are deemed significant.

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